Question 5d for BS3b, HT09: Statistical Lifetime Models

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Question: Suppose that instead of being known exactly, the initial number of chicks has been estimated at 100 ± 10. That is, the estimator is normal with mean 100 and standard deviation 10. Use the delta method to compute an approximate 95% confidence interval for q_0, the probability of dying in the first year.

Solution: The estimate is $\hat{q}_0 = D_0/N_0$, where D_0 is the observed number of deaths at age 0, and N_0 is the number initially under observation. We know that D_0 is approximately normal with mean $q_0 N_0$ and variance $\sigma_d^2 = q_0(1-q_0)N_0$. We write

$$
\begin{aligned}
Z_1 &= (N_0 - 100)/10 \\
Z_2 &= (D_0 - N_0 q_0)/\sigma_d.
\end{aligned}
$$

These are both approximately standard normal. Reordering, we have

$$
\begin{aligned}
N_0 &= 100(1 + \epsilon_1 Z_1), \\
D_0 &= N_0 q_0 \left(1 + \frac{\sigma_d}{N_0 q_0} Z_2\right) = N_0 q_0 \left(1 + \epsilon_2 Z_2\right)
\end{aligned}
$$

where

$$
\epsilon_1 = 0.1, \qquad \epsilon_2 = \sqrt{\frac{1-q_0}{N_0 q_0}}.
$$

Observe that $\epsilon_2 \approx \sqrt{.72/28} = 0.16$. Since both ϵ_i are small, we may apply the delta method by performing a Taylor series expansion, and dropping terms beyond first order in ϵ:

$$
\begin{aligned}
\frac{D_0}{N_0} &= q_0 \left(1 + \epsilon_2 Z_2\right) \left(1 + \epsilon_1 Z_1\right)^{-1} \\
&\approx q_0 \left(1 - \epsilon_1 Z_1 + \epsilon_2 Z_2\right).
\end{aligned}
$$

Consequently, $\hat{q}_0$ is approximately normal, with mean q_0 and variance $q_0^2(\epsilon_1^2 + \epsilon_2^2) \approx 0.0028$. So the standard error of the estimate is about 0.053, making an approximate 95% normal confidence interval

$$
\hat{q}_0 \pm 1.96 \cdot 0.053 = (.176, .384).
$$