Lecture 8

Multiple Decrements: Theory and Examples

8.1 Estimation for general multiple decrements

We can deduce from either description in the previous section that the likelihood for a sample of n independent lifetimes $y_1, \ldots, y_n$ and respective new states $j_1, \ldots, j_n$, each (y_i, j_i) sampled from (L, J), is given by

$$\prod_{i=1}^{n} \lambda_{j_i}(y_i) \exp\left\{-\int_0^{y_i} \lambda_+(t)dt\right\}.$$

Let us assume that the forces of decrement $\lambda_j(t) = \lambda_j(x + \frac{1}{2})$ are constant on $x \le t < x + 1$, for all $x \in \mathbb{N}$ and $1 \le j \le m$. Then the likelihood can be given as

$$\prod_{x \in \mathbb{N}} \prod_{j=1}^{m} \left(\lambda_j(x + \tfrac{1}{2})\right)^{d_{j,x}} \exp\left\{-\tilde{\ell}_x \lambda_j(x + \tfrac{1}{2})\right\},$$

where $d_{j,x}$ is the number of decrements to state j between ages x and $x + 1$, and $\tilde{\ell}_x$ is the total time spent alive between ages x and $x + 1$.

Now the parameters are $\lambda_j(x + \frac{1}{2})$, $x \in \mathbb{N}$, $1 \le j \le m$, and they are again well separated to deduce

$$\hat{\lambda}_j(x + \tfrac{1}{2}) = \frac{d_{j,x}}{\tilde{\ell}_x}, \qquad 1 \le j \le m, \quad 0 \le x \le \max\{L_1, \ldots, L_n\}.$$

Similarly, we can try to adapt the method to get maximum likelihood estimators from the curtate lifetimes. We can write down the likelihood as

$$\prod_{i=1}^{n} p_{(J,K)}(j_i, [y_i]) = \prod_{x \in \mathbb{N}} (1 - q_x)^{\ell_x - d_x} \prod_{j=1}^{m} q_{j,x}^{d_{j,x}},$$

but $1 - q_x = 1 - q_{1,x} - \ldots - q_{m,x}$ does not factorise, so we have to maximise simultaneously for all $1 \le j \le m$ expressions of the form

$$(1 - q_1 - \ldots - q_m)^{\ell - d_1 - \ldots - d_m} \prod_{j=1}^{m} q_j^{d_j}.$$

(We suppress the indices x.) A zero derivative with respect to q_j amounts to

$$(\ell - d_1 - \ldots - d_m)q_j = d_j(1 - q_1 - \ldots - q_m), \qquad 1 \le j \le m,$$

and summing over j gives

$$(\ell - d)q = d(1 - q) \qquad \Rightarrow \qquad q = \frac{d}{\ell}.$$

and then

$$(\ell - d)q_j = d_j(1 - q) \qquad \Rightarrow \qquad q_j = \frac{d_j(1-q)}{\ell - d} = \frac{d_j}{\ell}$$

so that, if we display the suppressed indices x again,

$$\hat{q}^{(0)}_{j,x} = \hat{q}^{(0)}_{j,x}(y_1, j_1, \ldots, y_n, j_n) = \frac{d_{j,x}}{\ell_x}.$$

Now we've done essentially all maximum likelihood calculations. This one was the only one that was not totally trivial. At repeated occurrences of the same factors, we have been and will be less explicit about these calculations. We'll derive likelihood functions, note that they factorise and identify the factors as being of one of the three forms

$$\begin{array}{rcl} (1-q)^{\ell-d}q^d & \Rightarrow & \hat{q} = d/\ell \\ \mu^d e^{-\mu\ell} & \Rightarrow & \hat{\mu} = d/\ell \\ (1 - q_1 - \ldots - q_m)^{\ell - d_1 - \ldots - d_m} \displaystyle\prod_{j=1}^m q_j^{d_j} & \Rightarrow & \hat{q}_j = d_j/\ell, \quad j = 1, \ldots, m. \end{array}$$

and deduce the estimates.

8.2 Example: Workforce model

A company is modelling its workforce using the model

$$Q(t) = \begin{pmatrix} -\lambda(t) - \sigma(t) - \mu(t) & \lambda(t) & \sigma(t) & \mu(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with four states $\mathbb{S} = \{W, V, I, \Delta\}$, where W ='working', V ='left the company voluntarily', I ='left the company involuntarily' and Δ ='left the company through death'.

If we observe n_x people aged x, then

$$\hat{\lambda}_{x+\frac{1}{2}} = \frac{d_{x,V}}{\tilde{\ell}_x}, \quad \hat{\sigma}_{x+\frac{1}{2}} = \frac{d_{x,I}}{\tilde{\ell}_x}, \quad \hat{\mu}_{x+\frac{1}{2}} = \frac{d_{x,\Delta}}{\tilde{\ell}_x}$$

where $\tilde{\ell}_x$ is the total amount of time spent working aged x, $d_{x,V}$ is the total number of workers who left the company voluntarily aged x, $d_{x,I}$ is the total number of workers who left the company involuntarily aged x, $d_{x,\Delta}$ is the total number of workers dying aged x.