Lecture 9

Multiple decrements: The distribution of the endpoint

9.1 Which state do we end up in?

The time-homogeneous multiple-decrement model makes a transition at the minimum of m exponential clocks as opposed to one clock in the single decrement model. In the same way, we can construct the time-inhomogeneous multiple-decrement model from m independent clocks C_j with hazard function $\lambda_j(t)$, $1 \leq j \leq m$. Then the likelihood for a transition at time t to state j is the product of $f_{C_j}(t)$ and $\bar{F}_{C_k}(t)$.

By Exercise A.1.2, the hazard function of $L = \min\{C_1, \dots, C_m\}$ is given by $h_L(t) = h_{C_1}(t) + \dots + h_{C_m}(t) = \lambda_1(t) + \dots + \lambda_m(t) = \lambda_+(t)$, and we can also calculate

$$
\begin{aligned}
\mathbb{P}(L = C_j | L = t) &= \lim_{\varepsilon \downarrow 0} \frac{\mathbb{P}(t \leq C_j < t + \varepsilon, \min\{C_k : k \neq j\} \geq C_j)}{\mathbb{P}(t \leq L < t + \varepsilon)} \\
&\begin{cases} \geq & \lim_{\varepsilon \downarrow 0} \dfrac{\frac{1}{\varepsilon}\mathbb{P}(t \leq C_j < t + \varepsilon, \min\{C_k : k \neq j\} \geq t + \varepsilon)}{\frac{1}{\varepsilon}\mathbb{P}(t \leq L < t + \varepsilon)} \\ \leq & \lim_{\varepsilon \downarrow 0} \dfrac{\frac{1}{\varepsilon}\mathbb{P}(t \leq C_j < t + \varepsilon, \min\{C_k : k \neq j\} \geq t)}{\frac{1}{\varepsilon}\mathbb{P}(t \leq L < t + \varepsilon)} \end{cases} \\
&= \frac{f_{C_j}(t) \prod_{k \neq j} \bar{F}_{C_k}(t)}{f_L(t)} = \frac{h_{C_j}(t)}{h_L(t)} = \frac{\lambda_j(t)}{\lambda_+(t)},
\end{aligned}
$$

and we obtain

$$
\mathbb{P}(L = C_j) = \int_0^\infty \mathbb{P}(L = C_j | L = t) f_L(t) dt = \int_0^\infty \lambda_j(t) \bar{F}_L(t) dt = \mathbb{E}(\Lambda_j(L)), \qquad (1)
$$

where $\Lambda_j(t) = \int_0^t \lambda_j(s) ds$ is the integrated hazard function. (For the last step we used that $\mathbb{E}(g(L)) = \int_0^\infty g'(t) \bar{F}_L(t) dt$ for all increasing differentiable $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$.)

The discrete (curtate) lifetime model: We can also split the curtate lifetime $K = [L]$ according to the type of decrement J ($J = j$ if $L = T_j$) and define

$$
q_{j,x} = \mathbb{P}(L < x + 1, J = j | L > x), \qquad 1 \leq j \leq m, \quad x \in \mathbb{N},
$$

then clearly for $x \in \mathbb{N}$

$$
q_{1,x} + \dots + q_{m,x} = q_x
$$

and, for $1 \leq j \leq m$,

$$p_{(J,K)}(j,x) = \mathbb{P}(J = j, K = x) = \mathbb{P}(L \leq x+1, J = j | L > x)\mathbb{P}(L > x) = p_0 \dots p_{x-1} q_{j,x}.$$

Note that this bivariate probability mass function is simple, whereas the joint distribution of (L, J) is conceptually more demanding since L is continuous and J is discrete. We chose to express the marginal probability density function of L and the conditional probability mass function of J given $L = t$. In the assignment questions, you will see an alternative description in terms of sub-probability densities $g_j(t) = \frac{d}{dt}\mathbb{P}(L \leq t, J = j)$, which you can normalise – $g_j(t)/\mathbb{P}(J = j)$ is the conditional density of L given $J = j$.

9.2 Cohabitation dissolution model

There has been considerable interest in the influence of nonmarital birth on the likelihood of a child growing up without one of its parents. In the paper [Kie01] relevant data are given for nine different western European countries. We give a summary of some of the UK data in Table 9.1. We represent the data in terms of a multiple decrement model in which the one nonabsorbing state is cohabitation, and this leads to the two absorbing states, which are marriage or separation. (Of course, there is a third absorbing state, corresponding to the death of one of the partners, but this did not appear in the data. And of course, the marriage state is not actually absorbing, except in fairy tales. A more complete analysis would splice this model onto a model of the fate of marriages.) Time, in this model, begins with the birth of the first child. Because of the way the data are given, we treat the hazard rates as constant in the time intervals $[0, 1]$, $[1, 3]$, and $[3, 5]$. There are no data about what happens after 5 years. We write d_x^M and d_x^S for the number of individuals marrying and separating, respectively, and similar for the estimation of hazard rates. (For simplicity, we have divided the separation data, which were actually only given for the periods $[0, 3]$ and $[3, 5]$, as though there were a count for separations in $[0, 1]$.)

Table 9.1: Data from [Kie01] on rates of conversion of cohabitations into marriage or separation, by years since birth of first child

(a) % cohabiting couples remaining together (from among those who did not marry)

n	after 3 years	after 5 years
106	61	48

(b) % of cohabiting couples who marry within stated time.

n	1 year	3 years	5 years
150	18	30	39

Translating the data in Table 9.1 into a multiple-decrement life table requires some interpretive work.

1. There are only 106 individuals given for the data on separation; this is because the individuals who eventually married were excluded from this tabulation.

2. The data are given in percentages.

3. There is no count of separations in the first year.

4. Note that separations are given by survival percentages, while marriages are given by loss percentages.

We now construct a combined life table from the data in Table 9.1. The purpose of this model is to integrate information from the two data sets. This requires some assumptions, to wit, that the transition rates to the two different absorbing states are the same for everyone, and that they are constant over the periods 0–1, 1–3, 3–5 (and constant over 0–3 for separation).

The procedure is essentially the same as the construction of the single-decrement life table, except that the survival is decremented by both loss counts d_x^M and d_x^S; and the estimation of years at risk $\tilde{\ell}_x$ now depends on both decrements, so is

$$\tilde{\ell}_x = \ell_{x'}(x' - x) + (d_x^M + d_x^S)\frac{x' - x}{2},$$

where x' is the next age on the life table. Thus, for example, $\tilde{\ell}_1$, which is the number of years at risk from age 1 to 3, is $64 \cdot 2 + 41 \cdot 1 = 169$.

One of the challenges is that we observe transitions to Separation conditional on never being Married, but the transitions to Married are unconditioned. We begin by computing a single-decrement life table for Separation. This is quite straightforward, since we have a cohort under (presumably) complete observation. We start with $\ell_0 = 106$, and the decrements in the ensuing time period are $d_0 = 106 \times 0.39 = 41$. We estimate the total years under observation as $\tilde{\ell}_0 \approx 106 \times 3 - 41 \times 3/2 = 256.5$. From these numbers we compute our central estimate $\hat{\mu}_0^S = 41/256 = 0.160$. Carrying through the same calculations to the time-period 3–5, we get the results in Table 9.2.

Table 9.2: Single decrement table of cohabiting relationships, subject to ending only by separation, computed from data in Table 9.1(a).

x	ℓ_x	d_x	$\tilde{\ell}_x$	$\hat{\mu}_x^S$
0–3	106	41	256.5	0.160
3–5	65	14	116	0.121

We then use the rates computed for separation, and apply them to build a multiple decrements table. In one respect, the data for marriage are more straightforward: These are absolute decrements, rather than conditional ones. If we set up a life table an a radix of 1000, we know that the decrements due to marriage should be exactly the percentages given in Table 9.1(b); that is, 180, 120, and 90. We put these into column 2 of our multiple-decrements life table, Table 9.3. We can also fill in the decrement rates μ_x^S, which have already been computed.

Our goal is to compute μ_x^M. We know the number of marriages, but we still need to estimate $\tilde{\ell}_x$, the total number of years at risk, in each age class. This is the one slightly tricky point in this calculation. We can approximate

$$\begin{aligned}\tilde{\ell}_x &\approx (x' - x)\ell_x - \frac{1}{2}(x' - x)(d_x^S + d_x^M)\\ &\approx (x' - x)\ell_x - \frac{1}{2}(x' - x)(\mu_x^S\tilde{\ell}_x + d_x^M), \text{ so that}\\ \tilde{\ell}_x &\approx \frac{(x' - x)[\ell_x - d_x^M/2]}{1 + (x' - x)\mu_x^S/2}.\end{aligned}$$

Table 9.3: Multiple decrement life table for survival of cohabiting relationships, from time of birth of first child, computed from data in Table **??**.

x	ℓ_x	d_x^M	d_x^S	$\tilde{\ell}_x$	$\hat{\mu}_x^M$	$\hat{\mu}_x^S$
0–1	1000	180	135	843	0.214	0.160
1–3	685	120	173	1078	0.111	0.160
3–5	13	90	75	644	0.134	0.121

Substituting in the values we already know, we get

$$\tilde{\ell}_0 \approx \frac{1000 - 90}{1.08} = 843,$$

so that $\hat{\mu}_x^M \approx 180/843 = 0.214$. This completes the first row of the table, and by the same methods we complete Table 9.3. (We compute $d_x^S = \mu_x^S \tilde{\ell}_x$.)

We are now in a position to use the model to draw some potentially interesting conclusions. For instance, we may be interested to know the probability that a cohabitation with children will end in separation. We need to decide what to do with the lack of observations after 5 years. For simplicity, let us assume that rates remain constant after that point, so that all cohabitations would eventually end in one of these fates. Applying the formula (1), we see that

$$\mathbb{P}\{\text{separate}\} = \int_0^\infty \mu_x^S \bar{F}(x) dx.$$

We have then

$$\begin{aligned}
\mathbb{P}\{\text{separate}\} &= 0.160 \int_0^1 e^{-0.374x} dx + 0.160 \int_1^3 e^{-0.374-0.325(x-1)} dx + 0.121 \int_3^\infty e^{-0.924-0.255(x-3)} dx \\
&= \frac{0.160}{0.374}\left[1 - e^{-0.374}\right] + \frac{0.160}{0.325}\left[e^{-0.374} - e^{-0.924}\right] + \frac{0.121}{0.255}\left[e^{-0.924}\right] \\
&= 0.133 + 0.143 + 0.188 \\
&= 0.464.
\end{aligned}$$