A.4 Multiple decrements and general Markov models

Please hand in your work by Monday, 22 February 2010, 4pm, at the Department of Statistics.

1. A life office uses the three-state healthy-sick-dead model in the pricing of its long term sickness policies. The transition rates are assumed to be constant. Denote the state space by $\mathbb{S} = \{H, S, \Delta\}$ and transition rates by $\sigma = q_{HS}$, $\rho = q_{SH}$, $\mu = q_{H\Delta}$, $\nu = q_{S\Delta}$.

   For a group of policy holders, over a one-year period the following data were recorded:

transition from	number
H to S	15
H to Δ	6
S to H	5
S to Δ	1

   The total times spent in states H and S were 625 years and 35 years, respectively.

   (a) Write down the likelihood function for this model and show that this is maximized when $\sigma = 0.024$.

   (b) Determine the asymptotic distribution of $\hat{\sigma}$, the MLE of σ.

   (c) Calculate an estimate of the standard deviation of $\hat{\sigma}$.

   (d) Construct an approximate 95% confidence interval for σ.

   (e) The policyholder pays contributions at rate C when in state H and receives benefits at rate B when in state S. No death benefit is payable. The life office uses the model to set the ratio of contributions to benefits. Briefly explain how this can be done.

   (f) A trainee believes that the model is too simplistic. For each of the trainee's suggestions below, comment on whether following the suggestion would be likely to improve the model's predictive power.

      i. The transition rates should depend on the age of the policyholder.

      ii. The transition rates should vary according to the time of year.

      iii. ρ and ν should also depend on the duration of the sickness to date.

   Discuss briefly which suggestions you could carry out, in principle, with techniques you have learned in the lectures. What additional data do you need?

   Outline the principal difficulty in fitting a model with parameters dependent on all these factors.

2. Consider a single-server queueing system.

   (a) Denote the arrival rate by λ, the service rate by μ. Starting from an empty system and given observations up to the nth transition,

      i. write down the likelihood function and determine the maximum likelihood estimator $(\hat{\lambda}, \hat{\mu})$;

      ii. calculate the bivariate asymptotic distribution of $(\hat{\lambda}, \hat{\mu})$. Deduce that $\hat{\lambda}$ and $\hat{\mu}$ are asymptotically independent.

      iii.* derive separate approximate $(1-\alpha)$-CIs for λ and μ, and joint approximate $(1-\alpha)$-confidence regions, either rectangular or with minimal area.

   (b) Suppose that the queue length cannot increase beyond m and the length of the queue has an impact on the arrival rates (but not on the service times).

i. How do you model this situation?

ii. Derive maximum likelihood estimators.

iii. Suppose $m = 2$. If $\lambda_0 = \lambda_1$, what is the large-sample distribution of

$$\frac{\hat{\lambda}_0 - \hat{\lambda}_1}{\sqrt{\mathrm{Var}(\hat{\lambda}_0) + \mathrm{Var}(\hat{\lambda}_1)}} \quad \text{or} \quad \frac{(\hat{\lambda}_0 - \hat{\lambda}_1)^2}{\mathrm{Var}(\hat{\lambda}_0) + \mathrm{Var}(\hat{\lambda}_1)}?$$

iv.* By estimating the variance, deduce a test for $H_0 : \lambda_0 = \lambda_1$ vs $H_1 : \lambda_0 \neq \lambda_1$.

v.* Generalise iii. and iv. to $m \geq 3$. *(* = optional)*

3. Public health officials often compare the effects of different changes in population health by considering the resulting change in life expectancy. But what about the personal and societal costs of illness and disability? This has led to the notion of "Disability Adjusted Life Years" (DALYs) or "Quality Adjusted Life Years" (QALYs), in which years are weighted by some measure of the "quality of life". Suppose we have a Markov representation of lifetimes, in which there are m non-absorbing "alive" states, and 1 absorbing "dead" state. The "quality" of a year of life in state i is w_i. Individuals begin in state i with probability p_i.

   (a) Show that the expected total value of a life is given by $p^T Q_*^{-1} w$, where T means "transpose", and Q_* is the submatrix of Q corresponding to the non-absorbing states.

   (b) Suppose the model is the simple Healthy-Sick-Dead model described in section 10.4.2, with $\sigma = 0.1$, $\rho = 0$, $\delta = 0.01$, and $\gamma = 0.2$. What is the life expectancy of someone initially healthy?

   (c) Suppose we judge a year of being sick to be worth half of a year of health. What is the expected number of quality-adjusted years in the lifetime of someone initially healthy? What about the number remaining to someone who has just gotten sick?

   (d) How many QALYs would be saved by a cure that now raises ρ to 0.2? Suppose the alternative were a treatment that would lower γ by some amount. Could you achieve the same QALY savings by such a treatment? How low would γ have to go?

   (e) What fraction of individuals are sick right before they die?

   (f) Suppose after 30 years the survivors move into "old age", in which the rate of becoming sick σ rises to 0.5. What is the effect on QALYs?