# Lecture 3

# Probability II

## 3.1 Independence and the Multiplication Law

**??** If the probability that one event A occurs doesn't affect the probability that the event B also occurs, then we say that A and B are **independent**. For example, it seems clear than one coin doesn't know what happened to the other one (and if it did know, it wouldn't care), so if $A_1$ is the event that the first coin comes up heads, and $A_2$ the event that the second coin comes up heads, then

### Example 3.1: One die, continued

Continuing from Example 2.3, with event $A_1 = \{\text{face is even}\} = \{2, 4, 6\}$ and $A_2 = \{\text{face is greater than } 4\} = \{5, 6\}$, we see that $A_1 \cap A_2 = \{6\}$

$$\begin{aligned} P(A_1) &= \frac{3}{6} = 0.5, \\ P(A_2) &= \frac{2}{6} = 0.33, \\ P(A_1 \cap A_2) &= \frac{1}{6} = P(A_1) \times P(A_2). \end{aligned}$$

On the other hand, if $A_3 = \{4, 5, 6\}$, then $A_3$ and $A_1$ are not independent. ■

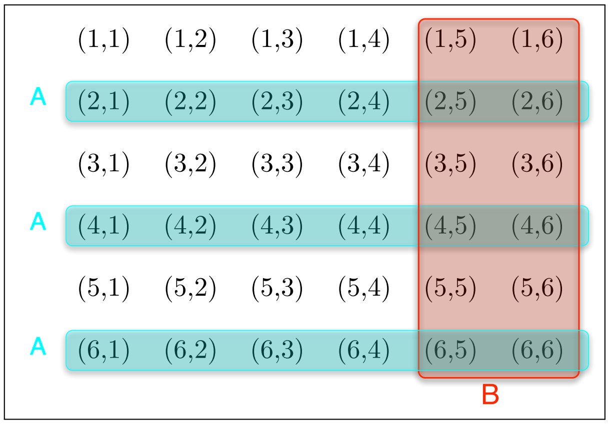
### Example 3.2: Two dice

Suppose we roll two dice. The sample space may be represented as pairs (first roll, second roll).

| | | | | | |
|---|---|---|---|---|---|
| (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) |
| (2,1) | (2,2) | (2,3) | (2,4) | (2,5) | (2,6) |
| (3,1) | (3,2) | (3,3) | (3,4) | (3,5) | (3,6) |
| (4,1) | (4,2) | (4,3) | (4,4) | (4,5) | (4,6) |
| (5,1) | (5,2) | (5,3) | (5,4) | (5,5) | (5,6) |
| (6,1) | (6,2) | (6,3) | (6,4) | (6,5) | (6,6) |

There are 36 points in the sample space. These are all equally likely. Thus, each point has probability 1/36. Consider the events

$$\begin{aligned} A &= \{\texttt{First roll is even}\},\\ B &= \{\texttt{Second roll is bigger than 4}\},\\ A \cap B &= \{\texttt{First roll is even and Second roll is bigger than 4}\}, \end{aligned}$$



| | | | | | |
|---|---|---|---|---|---|
| (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) |
| (2,1) | (2,2) | (2,3) | (2,4) | (2,5) | (2,6) |
| (3,1) | (3,2) | (3,3) | (3,4) | (3,5) | (3,6) |
| (4,1) | (4,2) | (4,3) | (4,4) | (4,5) | (4,6) |
| (5,1) | (5,2) | (5,3) | (5,4) | (5,5) | (5,6) |
| (6,1) | (6,2) | (6,3) | (6,4) | (6,5) | (6,6) |

Figure 3.1: Events $A = \{\texttt{First roll is even}\}$ and $B = \{\texttt{Second roll is bigger than 4}\}$.

We see from Figure 3.1 that $A$ contains 18 points and $B$ contains 12 points, so that $P(A) = 18/36 = 1/2$, and $P(B) = 12/36 = 1/3$. Meanwhile, $A \cap B$ contains 6 points, so $P(A \cap B) = 6/36 =$

$1/6 = 1/2 \times 1/3$. Thus A and B are independent. This should not be surprising: A depends only on the first roll, and B depends on the second. These two have no effect on each other, so the events must be independent. ■

This points up an important rule:

> Events that depend on experiments that can't influence each other are always independent.

Thus, two (or more) coin flips are always independent. But this is also relevant to analysing experiments such as those of Example 2.1. If the drug has no effect on survival, then events like {patient # 612 survived} are independent of events like {patient # 612 was allocated to the control group}.

### Example 3.3: Two dice

Suppose we roll two dice. Consider the events

$$\begin{aligned} A &= \{\texttt{First roll is even}\}, \\ C &= \{\texttt{Sum of the two rolls roll is bigger than 8}\}, \\ A \cap C &= \{\texttt{First roll is even and sum is bigger than 8}\}, \end{aligned}$$

Then we see from Figure 3.2 that $P(C) = 10/36$, and $P(A \cap C) = 6/36 \neq 10/36 \times 1/2$. On the other hand, if we replace $C$ by $D = \{\texttt{Sum of the two rolls roll is exactly 9}\}$, then we see from Figure 3.3 that $P(D) = 4/36 = 1/9$, and $P(A \cap D) = 2/36 = 1/9 \times 1/2$, so the events $A$ and $D$ are independent. We see that events may be independent, even if they are not based on separate experiments.

■

### Example 3.4: DNA Fingerprinting

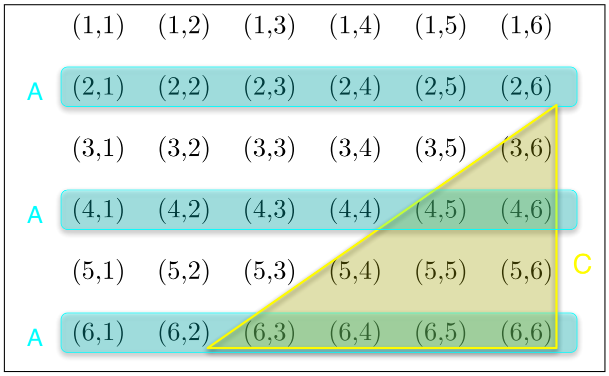


Figure 3.2: Events $A$ = {`First roll is even`} and $C$ = {`Sum is bigger than 8`}.

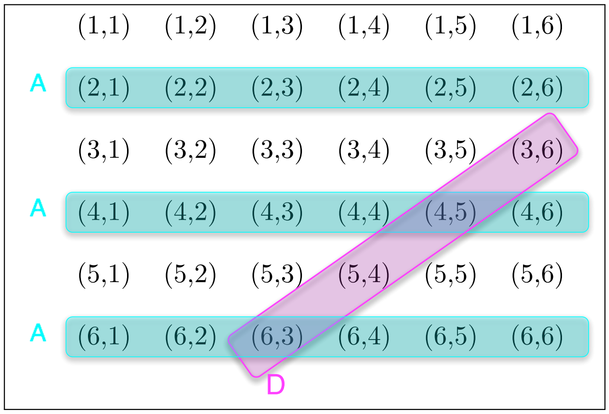


Figure 3.3: Events $A$ = {`First roll is even`} and $D$ = {`Sum is exactly 9`}.

A simplified description of how DNA fingerprinting works is this: The police find biological material connected to a crime. In the laboratory, they identify certain SNPs, finding out which version of each SNP the presumed culprit has. Then, they either search a database for someone who has the same versions of all the SNPs, or compares these SNPs to those of a suspect.

Searching a database can be potentially problematic. Imagine that the laboratory has found 12 SNPs, at which the crime-scene

> DNA has rare versions, each of which is found in only 10% of the population. They then search a database and find someone with all the same SNP versions. The expert then comes and testifies, to say that the probability of any single person having the same SNPs is $(1/10)^{12} = 1/1$ trillion. There are only 60 million people in the UK, so the probability of there being another person with the same SNPs is only about 60 million/1 trillion= 0.00006 — less than 1 in ten thousand. So it can't be mistaken identity.
>
> Except... Having particular variants at different SNPs are not independent events. For one thing, some SNPs in one population (Europeans, for example) may not be SNPs in other population (Asians, for example) where everyone may have the same variant. Thus, the 10% of the population that has each of these different rare SNP variants could in fact be the same 10%, and they may have all of these dozen variants in common because they all come from the same place, where everyone has just those variants.
>
> And don't forget to check whether the suspect has a monozygotic twin! More than 1 person in a thousand has one, and in that case, the twins will have all the same rare SNPs, because their genomes are identical. ■

## 3.2 Conditional Probability Laws

Suppose,

A = The event that a randomly selected student from the class has a bike
B = The event that a randomly selected student from the class has blue eyes

and P(A) = 0.36, P(B) = 0.45 and P(A∩B) = 0.13

What is the probability that a student has a bike **GIVEN** that the student has blue eyes?

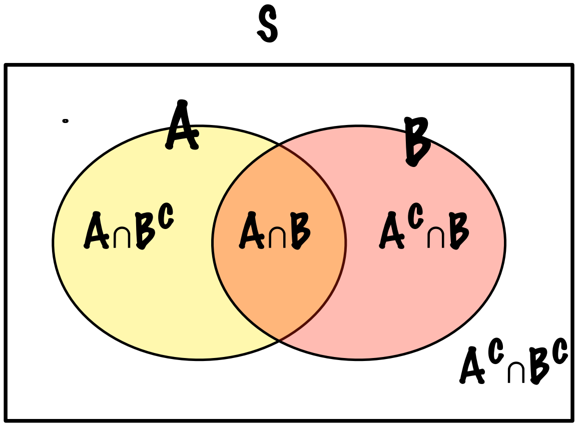
in other words

Considering just students who have blue eyes, what is the probability that a randomly selected student has a bike?

This is a **conditional** probability.

We write this probability as P(B|A) (pronounced 'probability of B given A')

Think of P(B|A) as 'how much of A is taken up by B'.



Then we see that

$$\boxed{\mathrm{P(B|A)} = \frac{\mathrm{P(A \cap B)}}{\mathrm{P(A)}}} \qquad \text{(Conditional Probability Law)}$$

### Example 3.5: SNPs again

We return to the setting of Example 2.5. What is the probability that a SNP is variable in the African population given that it is variable in the Asian population?

We have that

$$\begin{aligned} P(A) &= 0.7 \\ P(B) &= 0.8 \\ P(A \cap B) &= 0.6. \end{aligned}$$

We want

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.6}{0.8} = 0.75$$

■

We can rearrange the conditional probability law to obtain a general Multiplication Law.

$$\mathrm{P(B|A)} = \frac{\mathrm{P(A \cap B)}}{\mathrm{P(A)}} \Rightarrow \mathrm{P(B|A)P(A) = P(A \cap B)}$$

Similarly

$$\mathrm{P(A|B)P(B)} = \mathrm{P(A \cap B)}$$

$$\Rightarrow \boxed{\mathrm{P(A \cap B) = P(B|A)P(A) = P(A|B)P(B)}}$$

**Example 3.6: Multiplication Law**

If P(B) = 0.2 and P(A|B) = 0.36 what is P(A∩B)?

P(A∩B) = 0.36×0.2 = 0.072 ■

### 3.2.1 Independence of Events

**Definition** Two events are $A$ and $B$ are said to be *independent* if $P(A \cap B) = P(A)P(B)$.

Note that in this case (provided $P(B) > 0$), if $A$ and $B$ are independent

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A),$$

and similarly $P(B|A) = P(B)$ (provided $P(A) > 0$).

So for independent events, knowledge that one of the events has occurred does not change our assessment of the probability that the other event has occur.

### Example 3.7: Snails

In a population of a particular species of snail, individuals exhibit different forms. It is known that 45% have a pink background colouring, while 55% have a yellow background colouring. In addition, 30% of individuals are striped, and 20% of the population are pink and striped.

1. Is the presence or absence of striping independent of background colour?

2. Given that a snail is pink, what is the probability that it will have stripes.

Define the events: $A$, $B$, that a snail has a pink, respectively yellow, background colouring, and $S$ for the event that is has stripes.

Then we are told $P(A) = 0.45$, $P(B) = 0.55$, $P(S) = 0.3$, and $P(A \cap S) = 0.2$.

For part (1), note that

$$0.2 = P(A \cap S) \neq 0.135 = P(A)P(S),$$

so the events $A$ and $S$ are not independent.

For part (2),

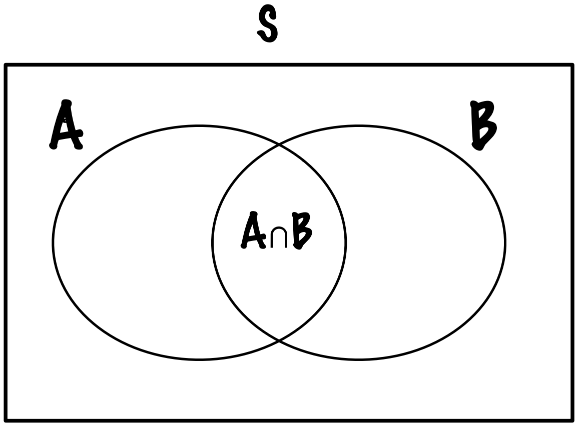
$$P(S|A) = \frac{P(S \cap A)}{P(A)} = \frac{0.2}{0.45} = 0.44.$$

Thus, knowledge that a snail has a pink background colouring increases the probability that it is striped. (That $P(S|A) \neq P(S)$ also establishes that background colouring and the presence of stripes, are not independent.) ■

### 3.2.2 The Partition law

The partition law is a very useful rule that allows us to calculate the probability of an event by spitting it up into a number of mutually exclusive events. For example, suppose we know that P(A∩B) = 0.52 and P(A∩B$^c$) = 0.14 what is p(A)?

P(A) is made up of two parts (i) the part of A contained in B (ii) the part of A contained in B$^c$.



So we have the rule

$$\boxed{\mathrm{P(A) = P(A \cap B) + P(A \cap B^c)}}$$

and P(A) = P(A∩B) + P(A∩B$^c$) = 0.52 + 0.14 = 0.66

More generally, if $E_1, \ldots, E_n$ are a set of *mutually exclusive* events then

$$\boxed{\mathrm{P(A)} = \sum_{i=1}^{n} \mathrm{P(A \cap E}_i) = \sum_{i=1}^{n} \mathrm{P(A| \; E}_i)\mathrm{P(E}_i)}$$

A set of events are *mutually exclusive* if at most one of the events can occur in a given experiment.

**Example 3.8: Mendelian segregation**

At a particular locus in humans, there are two alleles $A$ and $B$, and it is known that the population frequencies of the genotypes $AA$, $AB$, and $BB$, are 0.49, 0.42, and 0.09, respectively. An $AA$ man has a child with a woman whose genotype is unknown.

What is the probability that the child will have genotype $AB$?

We assume that as far as her genotype at this locus is concerned the woman is chosen randomly from the population.

Use the partition rule, where the partition corresponds to the three possible genotypes for the woman. Then

$$\begin{aligned}
P(\text{child } AB) &= P(\text{child } AB \text{ and mother } AA) \\
&\qquad +P(\text{child } AB \text{ and mother } AB) \\
&\qquad\quad +P(\text{child } AB \text{ and mother } BB) \\
&= P(\text{mother } AA)P(\text{child } AB|\text{mother } AA) \\
&\qquad +P(\text{mother } AB)P(\text{child } AB|\text{mother } AB) \\
&\qquad\quad +P(\text{mother } BB)P(\text{child } AB|\text{mother } BB) \\
&= 0.49 \times 0 + 0.42 \times 0.5 + 0.09 \times 1 \\
&= 0.3.
\end{aligned}$$

■

## 3.3 Bayes' Rule

One of the most common situations in science is that we have some observations, and we need to figure out what state of the world is likely to have produced those observations. For instance, we observe that a certain number of vaccinated people contract polio, and a certain number of unvaccinated people contract polio, and we need to figure out how effective the vaccine is. The problem is, our theoretical knowledge goes in the wrong direction: If the vaccine is so effective, how many people will contract polio. Bayes' Rule allows us to turn the inference around.

### Example 3.9: Medical testing

In a medical setting we might want to calculate the probability that a person has a disease D given they have a specific symptom

S, i.e. we want to calculate $P(D|S)$. This is a hard probability to assign as we would need to take a random sample of people from the population with the symptom.

A probability that is much easier to calculate is $P(S|D)$, i.e. the probability that a person with the disease has the symptom. This probability can be assigned much more easily as medical records for people with serious diseases are kept.

The power of Bayes Rule is its ability to take $P(S|D)$ and calculate $P(D|S)$.

*We have already seen a version of Bayes' Rule*

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

*Using the Multiplication Law we can rewrite this as*

$$\boxed{P(B|A) = \frac{P(A|B)P(B)}{P(A)}} \qquad \text{(Bayes Rule)}$$

Suppose $P(S|D) = 0.12$, $P(D) = 0.01$ and $P(S) = 0.03$. Then

$$P(D|S) = \frac{0.12 \times 0.01}{0.03} = 0.04$$

■

### Example 3.10: Genetic testing

A gene has two possible types $A_1$ and $A_2$. 75% of the population have $A_1$. $B$ is a disease that has 3 forms $B_1$ (mild), $B_2$ (severe) and $B_3$ (lethal). $A_1$ is a protective gene, with the probabilities of having the three forms given $A_1$ as 0.9, 0.1 and 0 respectively. People with $A_2$ are unprotected and have the three forms with probabilities 0, 0.5 and 0.5 respectively.

What is the probability that a person has gene $A_1$ given they have the severe disease?

The first thing to do with such a question is 'decode' the information, i.e. write it down in a compact form we can work with.

$\mathrm{P}(A_1) = 0.75 \qquad \mathrm{P}(A_2) = 0.25$

$\mathrm{P}(B_1|A_1) = 0.9 \qquad \mathrm{P}(B_2|A_1) = 0.1 \qquad \mathrm{P}(B_3|A_1) = 0$

$\mathrm{P}(B_1|A_2) = 0 \qquad \mathrm{P}(B_2|A_2) = 0.5 \qquad \mathrm{P}(B_3|A_2) = 0.5$

We want $\mathrm{P}(A_1|B_2)$?

From Bayes Rule we know that

$$\mathrm{P}(A_1|B_2) = \frac{\mathrm{P}(B_2|A_1)\mathrm{P}(A_1)}{\mathrm{P}(B_2)}$$

We know $\mathrm{P}(B_2|A_1)$ and $\mathrm{P}(A_1)$ but what is $\mathrm{P}(B_2)$?

We can use the Partition Law since $A_1$ and $A_2$ are mutually exclusive.

$$\begin{aligned}
\mathrm{P}(B_2) &= \mathrm{P}(B_2 \cap A_1) + \mathrm{P}(B_2 \cap A_2) && \text{(Partition Law)} \\
&= \mathrm{P}(B_2|A_1)\mathrm{P}(A_1) + \mathrm{P}(B_2|A_2)\mathrm{P}(A_2) && \text{(Multiplication Law)} \\
&= 0.1 \times 0.75 + 0.5 \times 0.25 \\
&= 0.2
\end{aligned}$$

We can use Bayes Rule to calculate $\mathrm{P}(A_1|B_2)$.

$$\mathrm{P}(A_1|B_2) = \frac{0.1 \times 0.75}{0.2} = 0.375$$

■

## 3.4 Probability Laws

$$\boxed{\mathrm{P}(\mathrm{A}^c) = 1 - \mathrm{P}(\mathrm{A})} \qquad \text{(Complement Law)}$$

$$\boxed{P(A\cup B) = P(A) + P(B) - P(A\cap B)}$$ (Addition Law)

$$\boxed{P(B|A) = \frac{P(A\cap B)}{P(A)}}$$ (Conditional Probability Law)

$$\boxed{P(A\cap B) = P(B|A)P(A) = P(A|B)P(B)}$$ (Multiplication Law)

If $E_1, \ldots, E_n$ are a set of *mutually exclusive* events then

$$\boxed{P(A) = \sum_{i=1}^{n} P(A\cap E_i) = \sum_{i=1}^{n} P(A|\ E_i)P(E_i)}$$ (Partition Law)

$$\boxed{P(B|A) = \frac{P(A|B)P(B)}{P(A)}}$$ (Bayes Rule)

## 3.5 Permutations and Combinations (Probabilities of patterns)

In some situations we observe a specific pattern from a large number of possible patterns. To calculate the probability of the pattern we need to count the total number of patterns. This is why we need to learn about permutations and combinations.

### 3.5.1 Permutations of $n$ objects

Consider 2 objects A B

Q. How many ways can they be arranged? i.e. how many **permutations** are there?

A. 2 ways AB BA

Consider 3 objects A B C

Q. How many ways can they be arranged (permuted)?

A. 6 ways ABC ACB BCA BAC CAB CBA

Consider 4 objects A B C D

Q. How many ways can they be arranged (permuted)?

A. 24 ways

| | | | | | |
|---|---|---|---|---|---|
| ABCD | ABDC | ACBD | ACDB | ADBC | ADCB |
| BACD | BADC | BCAD | BCDA | BDAC | BDCA |
| CBAD | CBDA | CABD | CADB | CDBA | CDAB |
| DBCA | DBAC | DCBA | DCAB | DABC | DACB |

There is a pattern emerging here.

| No. of objects | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|
| No. of permutations | 2 | 6 | 24 | 120 | 720 | ... |

Can we find a formula for the number of permutations of $n$ objects?

A good way to think about permutations is to think of putting objects into boxes.

Suppose we have 5 objects. How many different ways can we place them into 5 boxes?

| | | | | |
|---|---|---|---|---|
| | | | | |

There are 5 choices of object for the first box.

| | | | | |
|---|---|---|---|---|
| 5 | | | | |

There are now only 4 objects to choose from for the second box.

| | | | | |
|---|---|---|---|---|
| 5 | 4 | | | |

There are 3 choices for the 3rd box, 2 for the 4th and 1 for the 5th box.

| | | | | |
|---|---|---|---|---|
| 5 | 4 | 3 | 2 | 1 |

Thus, the number of permutations of 5 objects is $5 \times 4 \times 3 \times 2 \times 1$.

In general, the number of permutations of $n$ objects is

$$n(n-1)(n-2)\dots(3)(2)(1)$$

We write this as $n!$ (pronounced ‘n factorial’). There should be a button on your calculator that calculates factorials.

### 3.5.2 Permutations of $r$ objects from $n$

Now suppose we have 4 objects and only 2 boxes. How many permutations of 2 objects when we have 4 to choose from?

There are 4 choices for the first box and 3 choices for the second box

$$\lfloor\, 4 \mid 3 \,\rfloor$$

So there are 12 permutations of 2 objects from 4. We write this as

$$^4P_2 = 12$$

We say there are $^nP_r$ permutations of $r$ objects chosen from $n$.

The formula for $^nP_r$ is given by

$$\boxed{\quad {}^nP_r \;=\; \frac{n!}{(n-r)!} \quad}$$

To see why this works consider the example above $^4P_2$.

Using the formula we get

$$^4P_1 \;=\; \frac{4!}{2!} = \frac{4\times 3\times 2\times 1}{2\times 1} = 4\times 3$$

### 3.5.3 Combinations of $r$ objects from $n$

Now consider the number of ways of choosing 2 objects from 4 when the order doesn't matter. We just want to count the number of possible **combinations**.

We know that there are 12 permutations when choosing 2 objects from 4. These are

| | | | | | |
|---|---|---|---|---|---|
| AB | AC | AD | BC | BD | CD |
| BA | CA | DA | CB | DB | DC |

Notice how the permutations are grouped in 2's which are the same combination of letters. Thus there are $12/2 = 6$ possible combinations.

AB AC AD BC BD CD

We write this as

$$^4C_2 = 6$$

We say there are $^nC_r$ combinations of $r$ objects chosen from $n$.

The formula for $^nC_r$ is given by

$$\boxed{^nC_r \quad = \quad \frac{n!}{(n-r)!r!}}$$

Another way of writing this formula that makes it clearer is

$$^nC_r \quad = \quad \frac{^nP_r}{r!}$$

Effectively this says we count the number of permutations of $r$ objects from $n$ and then divide by $r!$ because the $^nP_r$ permutations will occur in groups of $r!$ that are the same combination.

## 3.6 Worked Examples

(i). Four letters are chosen at random from the word RANDOMLY. Find the probability that all four letters chosen are consonants.

8 letters, 6 consonants, 2 vowels

P(all four are consonants) $= \frac{\text{\# of ways of choosing 4 consonants}}{\text{\# of ways of choosing 4 letters}}$

# of ways of choosing 4 consonants $= {}^6C_4 = \frac{6!}{4!2!} = 15$

# of ways of choosing 4 letters $= {}^8C_4 = \frac{8!}{4!4!} = 70$

$\Rightarrow$ P(all four are consonants) $= \frac{15}{70} = \frac{3}{14}$

(ii). A bag contains 8 white counters and 3 black counters. Two counters are drawn, one after the other. Find the probability of drawing one white and one black counter, in any order

(a) if the first counter is replaced

(b) if the first counter is not replaced

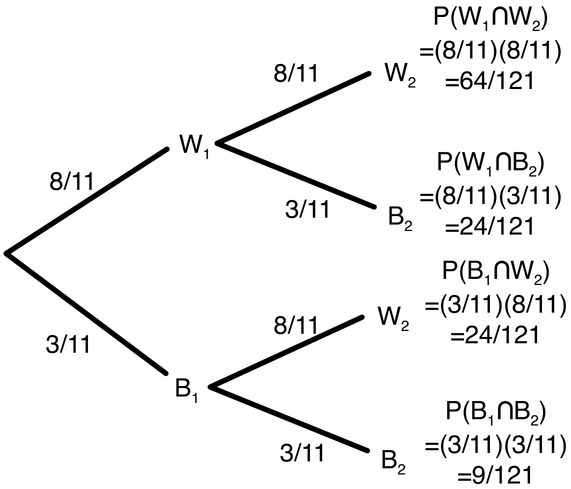
What is the probability that the second counter is black (assume that the first counter is replaced after it is taken)?

*A useful way of tackling many probability problems is to draw a 'probability tree'. The branches of the tree represent different possible events. Each branch is labelled with the probability of choosing it given what has occurred before. The probability of a given route through the tree can then be calculated by multiplying all the probabilities along that route (using the Multiplication Rule)*

(a) *With replacement*
Let
$W_1$ be the event 'a white counter is drawn first',
$W_2$ be the event 'a white counter is drawn second',
$B_1$ be the event 'a black counter is drawn first',
$B_2$ be the event 'a black counter is drawn second',

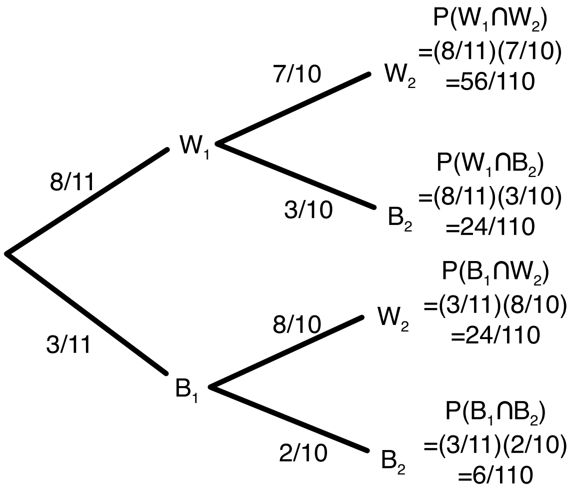
$$\begin{aligned} P(\text{one white and one black counter}) &= P(W_1 \cap B_2) + P(B_1 \cap W_2) \\ &= \frac{24}{121} + \frac{24}{121} \\ &= \frac{48}{121} \end{aligned}$$

(b) *Without replacement*



$$\begin{aligned} P(\text{one white and one black counter}) &= P(W_1 \cap B_2) + P(B_1 \cap W_2) \\ &= \frac{24}{110} + \frac{24}{110} \\ &= \frac{48}{110} \end{aligned}$$

$$\begin{aligned} P(\text{second counter is black}) &= P(W_1 \cap B_2) + P(B_1 \cap B_2) \\ &= \frac{24}{121} + \frac{9}{121} \\ &= \frac{33}{121} \end{aligned}$$



(iii). **From 2001 TT Prelim Q1** Two drugs that relieve pain are available to treat patients. Drug A has been found to be effective in three-quarters of all patients; when it is effective, the patients have relief from pain one hour after taking this drug. Drug B acts quicker but only works with one half of all patients: those who benefit from this drug have relief of pain after 30 mins. The physician cannot decide which patients should be prescribed which drug so he prescribes randomly. Assuming that there is no variation between patients in the times taken to act for either drug, calculate the probability that:

(a) a patient is prescribed drug B and is relieved of pain;

(b) a patient is relieved of pain after one hour;

(c) a patient who was relieved of pain after one hour took drug A;

(d) two patients receiving different drugs are both relieved of pain after one hour.

(e) out of six patients treated with the same drug, three are relieved of pain after one hour and three are not.

Let
$R30$ = The event that a patient is relieved of pain within 30 mins
$R60$ = The event that a patient is relieved of pain within 60 mins
$A$ = Event that a patient takes drug A
$B$ = Event that a patient takes drug B

$P(R60|A) = 0.75 \quad P(R30|B) = 0.5 \quad P(A) = P(B) = 0.5$

(a) $P(R30|B)P(B) = 0.25$

(b) $P(R60) = P(R60|A)P(A) + P(R60|B)P(B)$ since R30 ⇒ R60
$P(R60) = 0.75 \times 0.5 + 0.5 \times 0.5 = 0.625$

(c) $P(A|R60) = \frac{P(A \cap R60)}{P(R60)}$

$P(A \cap R60) = P(R60|A)P(A) = 0.75 \times 0.5 = 0.375$

$\Rightarrow P(A|R60) = \frac{0.375}{0.625} = 0.6$

(d) $P(R60|A)P(R60|B) = 0.75 \times 0.5 = 0.375$
Assuming the events are independent.

(e) $n = 6$ $X$ = no. of patients relieved of pain after 1hr

For $A$, $p = P(R60|A) = 0.75$

$P(X = 3|A) = {}^6C_3(0.75)^3(0.25)^3 = 0.1312$

For $B$, $p = P(R60|B) = 0.5$

$P(X = 3|B) = {}^6C_3(0.5)^3(0.5)^3 = 0.3125$

$$\begin{aligned}\Rightarrow P(X = 3) &= P(X = 3|A)P(A) + (X = 3|B)P(B) \\ &= 0.1312 \times 0.5 + 0.3125 \times 0.5 = 0.2222\end{aligned}$$

4. In the National Lottery you need to choose 6 balls from 49.

   What is the probability that I choose all 6 balls correctly?

   There are 2 ways of answering this question
   (i) using permutations and combinations
   (ii) using a tree diagram

### Method 1 - using permutations and combinations

$$\begin{aligned}
P(6 \text{ correct}) &= \frac{\text{No. of ways of choosing the 6 correct balls}}{\text{No. of ways of choosing 6 balls}} \\
&= \frac{{}^{6}P_{6}}{{}^{49}P_{6}} \\
&= \frac{\frac{6!}{0!}}{\frac{49!}{43!}} \\
&= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{49 \times 48 \times 47 \times 46 \times 45 \times 44} \\
&= 0.0000000715112 \quad (1 \text{ in } 14 \text{ million})
\end{aligned}$$

### Method 2 - using a tree diagram

Consider the first ball I choose, the probability it is correct is

$$\frac{6}{49}$$

The second ball I choose is correct with probability

$$\frac{5}{48}$$

The third ball I choose is correct with probability

$$\frac{4}{47}$$

and so on.

Thus the probability that I get all 6 balls correct is

$$\frac{6}{49}\frac{5}{48}\frac{4}{47}\frac{3}{46}\frac{2}{45}\frac{1}{44} = 0.0000000715112 \quad (1 \text{ in } 14 \text{ million})$$