A.3 Part A Probability Problem sheet 3: Markov chains

(1) Let $\mathcal{X} = \{1,2,3\}$, and let $X_1, X_2, \ldots$ be a Markov chain on $\mathcal{X}$ with transition matrix

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

What is $P(X_2 = 1 \,|\, X_0 = 1)$? $P(X_3 = 1 \,|\, X_0 = 3)$? Compute the stationary distribution of the chain.

(2) Identify the communicating classes of the Markov chains with the following transition matrices, and in each case determine which classes are closed:

$$\text{(i)} \quad \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \text{(ii)} \quad \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$

(3) A Markov chain on 4 states has the following transition matrix:

$$\begin{pmatrix} 0 & a & 0 & b \\ \frac{1}{2} & 0 & \frac{1}{3}+c & d \\ 0 & a & 0 & b \\ e & 0 & f & 0 \end{pmatrix}$$

Here a, b, c, d, e, f are nonnegative real numbers.

(a) What values of a, b, c, d, e, f are possible?

(b) What values of a, b, c, d, e, f make the Markov chain irreducible?

(c) What values of a, b, c, d, e, f make the Markov chain aperiodic?

(4) Show that two states in the same communicating class of a Markov chain must have the same period.

(5) A particle moves on the vertices of a cube (illustrated in figure 1) by moving at each step to one of its three nearest neighbours chosen uniformly at random. Let X_k be the distance from the starting point A (measured in graph distance — so, the number of edges in the shortest path) after k steps.

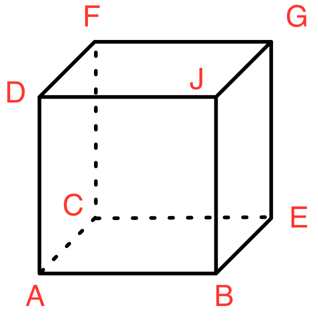

Figure 1

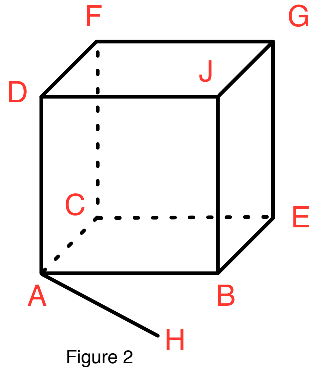

Figure 2

(a) Show that X_k is a Markov chain, and compute its transition matrix and stationary distribution.

(b) Compute the long-term average distance of the particle from the starting point.

(c) Suppose the particle never moves directly back along the edge it has just traversed. (That is, if the particle moves from edge x to edge y, its next step will not be directly back to x.) Show now that X_k is not a Markov chain.

(d) We add a tail onto the cube (labelled "H" in figure 2). Again, the particle moves to one of its nearest neighbours chosen uniformly at random, and let X_k be the distance from the starting point A. Is X_k a Markov chain?

(6) A Markov chain on finite state-space $\mathcal{X}$, with transition probabilities $P(x, y)$, is said to be **reversible** with **reversing distribution** π if $\pi_x P(x, y) = \pi_y P(y, x)$ for any $x, y \in \mathcal{X}$.

(a) Show that a reversing distribution is also a stationary distribution.

(b) Let

$$P = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Show that P is reversible, and find the reversing distribution.

(c) For a reversible ergodic Markov chain $X_1, X_2, \ldots$ show that

$$\lim_{t\to\infty} \mathbb{P}\{X_t = x, X_{t+1} = y\} = \lim_{t\to\infty} \mathbb{P}\{X_t = y, X_{t+1} = x\} \text{ for } x, y \in \mathcal{X}.$$

That is, the long-term "flow" from x to y is the same as the flow from y to x.

(d) Suppose we are given a set $\mathcal{X}$ and weights on pairs of states, which are a symmetric function $w : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$. (We can think of this as a graph with weighted edges; there is no edge between x and y where $w(x, y) = 0$.) Define transition probabilities by $P(x, y) = w(x, y)/\sum_{z\in\mathcal{X}} w(x, z)$. Show that this is reversible, and find the reversing measure.