Part A Probability

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Overview

The first half of the course takes further the probability theory that was developed in the first year. The aim is to build up a range of techniques that will be useful in dealing with mathematical models involving uncertainty. The second half of the course is concerned with Markov chains in discrete time and Poisson processes in one dimension, both with developing the relevant theory and giving examples of applications.

Synopsis

Continuous random variables. Jointly continuous random variables, independence, conditioning, bivariate distributions, functions of one or more random variables. Moment generating functions and applications. Characteristic functions, definition only. Examples to include some of those which may have later applications in Statistics. Basic ideas of what it means for a sequence of random variables to converge in probability, in distribution and in mean square. Chebychev and Markov inequalities. The weak law of large numbers and central limit theorem for independent identically distributed variables with a second moment. Statements of the continuity and uniqueness theorems for moment generating functions. Discrete-time Markov chains: definition, transition matrix, n-step transition probabilities, communicating classes, absorption, irreducibility, calculation of hitting probabilities and mean hitting times, recurrence and transience. Invariant distributions, mean return time, positive recurrence, convergence to equilibrium (proof not examinable). Examples of appli- cations in areas such as: genetics, branching processes, Markov chain Monte Carlo. Poisson processes in one dimension: exponential spacings, Poisson counts, thinning and superposition.

A word about stories and proofs

Probability is a highly theoretical area of mathematics, but also has many applications. We will spend a significant amount of time talking about some of the standard probability "stories". You need to learn these. For instance, you need to know that the binomial distribution is connected to the story of successes in independent trials with the same probability of success, and understand how to interpret that story in the context of a given problem.

At the same time, there are some deep mathematical concepts that need to be understood. Proofs are crucial to understanding these concepts. The question of whether a proof is "examinable" should not be uppermost in your mind. Proofs are tools for clarifying your understanding of how the assumptions of a theorem are linked to the conclusions, and why they are (or perhaps are not) essential.

To this end, there will be exercises that ask you to generalise standard proofs to new settings, in ways that will force you to grapple with the techniques. In many cases it will be clear that these proofs are too long to fit into the context of an exam paper. Some of you may be tempted to neglect these as not essential. I advise you to resist this temptation.

Exercises

There are four problem sheets, each covering two weeks of the course. There are no explicit review exercises, but you may want to look back at your Mods sheets, to make sure you still remember the material.

Doing problems is the most important way to learn a mathematical subject, and for no subject is that more important than for probability. Some important principles:

- Problems vary in difficulty. There are intended to be questions that will challenge the very best students in the course. An average student should find that there are multiple questions or parts of questions that he or she cannot do. Since questions are organised in part by topic, and since different students find different topics difficult, you will not necessarily find them to be ordered by difficulty. So if you can't do some questions, don't panic! Do what you can, think about the question a while, and plan to discuss it in tutorials.
- There is a variety of problems:
  - (i). Straightforward application of new techniques;
  - (ii). Extensions of techniques to novel situations;
  - (iii). Interpretation of applications, often involving extensive descriptions of an idealised real-world setting. These are particularly important in probability;
  - (iv). Proofs and theoretical exercises. See the above section.
- Some tutors will have you turn in questions ahead of the relevant lectures. This is not optimal, but seems unavoidable. If this is your situation, you'll have to do the best you can from reading.
- Problem sheets are not the same as exam questions. Problem sheets are part of the learning process; exam questions are merely for evaluation of your learning. Exam questions are designed to be done completely in no more than 40 minutes. You should be spending about 8 hours a week on each of your 16-hour lecture courses,

and most of that time will be spent doing exercises. Thus, 12–15 hours on a two-week sheet is not excessive. The hardest question on a sheet will be much harder, and take much longer to do, than the hardest exam question.

Reading

16 hours of lectures are not enough to cover all of the material in any depth. The lectures are intended to provide a framework for understanding what you will then learn in depth through reading and

Your reading should begin with the lecture notes, but should not end there. You will not be able to master the material by reading one source, and lecture notes are, by their nature, not worked out nearly as carefully as a book.

The official suggested supplemental reading, covering the entire syllabus:

- G. R. Grimmett and D. R. Stirzaker, Probability and Random Processes (3rd edition, OUP, 2001). Chapters 4, 6.1—6.5, 6.8.
- G. R. Grimmett and D. R. Stirzaker, One Thousand Exercises in Probability (OUP, 2001).
- G. R. Grimmett and D J A Welsh, Probability: An Introduction (OUP, 1986). Chapters 6, 7.4, 8, 11.1—11.3.
- J. R. Norris, Markov Chains (CUP, 1997). Chapter 1.
- D. R. Stirzaker, Elementary Probability (Second edition, CUP, 2003). Chapters 7—9 excluding 9.9.

These are some suggestions, but there is nothing compulsory about them. The material in this course is absolutely standard, and as a consequence there are hundreds of books covering it in different ways. You will learn best if you browse various books to find a style of presentation (and of thought) that makes the most sense to you. Here is a list of some other probability texts with my personal comments. This has **no official sanction**!

- Sheldon Ross — **A first course in probability theory**. Loads of good examples. Text a bit dry and plodding. His **Introduction to Probability Models** (now in its 9th edition) has some of the same pros and cons for Markov chains, Poisson process, and a lot more. The range of exercises is breathtaking.
- David Williams — **Weighing the odds**. A recent text by an inspiring author, with an integrated treatment of probability and statistics. Fast and exciting.
- Kai Lai Chung — **Elementary probability theory**. Slower exposition than the above. Good on counting.

- William Feller — **An introduction to probability theory and its applications**. The classic book, in 2 volumes. Volume 1 is just discrete distributions (including Markov chains), volume 2 is general probability, including convergence of distributions. Generations of probabilists have been inspired by this book, and it still can't be beat for its treatment of renewal processes, densities, and random walks. Deep exercises involving many interesting applications. Good treatment of convergence in distribution.
- Jim Pitman — **Probability**. Covers the more elementary parts of the course. Lots of entertaining examples worked through in the text, and exercises with solutions. Encyclopaedic array of exercises. Terrific text. Particularly good on conditional expectations and applying the normal approximation. Also great section on the bivariate normal distribution. Very intuitive treatment of the Poisson process
- Henk Tijms – **Understanding probability**. Full of anecdotal details – the first half is designed to be read as fun and to act as motivation for the mathematical details in the second half.
- Grinstead and Snell — **Introduction to Probability**. This is available in its entirety on-line. Long and thorough, but gentle.
- P. Billingsley — **Convergence of Probability Measures**. Above the level of this course, but this is the text to go to for those who are interested in really understanding how convergence of probability distributions really works.
- Kemeny and Snell — **Finite Markov Chains**. Another classic (1960), though the only thing that's outdated is the typeface. (Well, the applications are meager.)
- James Norris — **Markov Chains**. Officially recommended text of the course. Not bad. The section on discrete Markov chains is not all that long, and perhaps less accessible than it could be.